Asymptotic solitons of the extended Korteweg–de Vries equation

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The interaction of two higher-order solitary waves, governed by the extended Korteweg–de Vries (KdV) equation, is examined. A nonlocal transformation is used on the extended KdV equation to asymptotically transform it to the KdV equation. The transformation is used to derive the higher-order two-soliton collision and it is found that the interaction is asymptotically elastic. Moreover, the higher-order corrections to the phase shifts suffered by the solitary waves during the collision are found. Comparison is made with a previous result, which indicated that, except for a special case, the interaction of higher-order KdV solitary waves is inelastic, with a coupling, or interaction, term occuring after collision. It is shown that the two theories are asymptotically equivalent, with the coupling term representing the higher-order phase shift corrections. Finally, it is concluded, with the support of existing numerical evidence, that the interpretation of the coupling term as a higher-order phase shift is physically appropriate; hence, the interaction of higher-order solitary waves is asymptotically elastic. [S1063-651X(99)03803-5]

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I. INTRODUCTION

The Korteweg–de Vries (KdV) equation arises as an approximate equation governing weakly nonlinear long waves when terms up to first-order in the (small) wave amplitude are retained and the weakly nonlinear and weakly dispersive terms are in balance (see, Whitham in [2]). If higher-order effects are of interest then, by retaining terms up to second-order in the (small) wave amplitude, the extended KdV equation

$$\eta_t+6\eta\eta_x+\eta_{xxx}+\alpha c_1\eta^2\eta_x+\alpha c_2\eta_x\eta_{xx}+\alpha c_3\eta\eta_{xxx} + \alpha c_4\eta_{xxxxx}=0, \quad \alpha\ll 1 \tag{1}$$

results, where α is a nondimensional measure of the (small) wave amplitude and the coefficients c_i depend on the physical context. This equation describes the evolution of steeper waves of shorter wavelength than does the KdV equation. In the special case of surface waves on shallow water, Marchant and Smyth [3] found the coefficients to be

$$c_1=-1, \quad c_2=\tfrac{23}{6}, \quad c_3=\tfrac{5}{3}, \quad c_4=\tfrac{19}{60}. \tag{2}$$

Kodama [4] obtained an approximate Hamiltonian for the extended KdV equation (1). The Hamiltonian is exact for the integrable version of the extended KdV equation, with coefficients

$$c_1=1, \quad c_2=\tfrac{2}{3}, \quad c_3=\tfrac{1}{3}, \quad c_4=\tfrac{1}{30}, \tag{3}$$

and accurate to $O(\alpha)$ in the general case. The Hamiltonian system was transformed, to $O(\alpha)$, to the integrable case, which implies that the extended KdV equation with arbitrary coefficients is approximately integrable. The asymptotic transformation used included a nonlocal term.

Kraenkel [1] applied the nonlocal asymptotic transformation, as used by Kodama [4], to transform the two-soliton solution of the associated higher-order integrable KdV equation [(1) with coefficients (3)] to the higher-order two-soliton solution of the extended KdV equation (1) with arbitrary coefficients. The higher-order two-soliton solution contained interaction terms, which were interpreted as evidence of an inelastic collision. One special case was identified, when $c_3=10c_4$, for which the collision was elastic.

Marchant and Smyth [5] also transformed the extended KdV equation, Eq. (1) with arbitrary coefficients, to the associated higher-order integrable KdV equation. In contrast to Kraenkel [1] and Kodama [4], a local asymptotic transformation was used. The higher-order two-soliton solution for Eq. (1) was considered well after interaction and found to consist of two higher-order solitary waves of Eq. (1), which were unchanged in shape. Moreover, no dispersive radiation was present; hence, it was concluded that the collision was elastic to at least $O(\alpha)$. The $O(\alpha)$ corrections to the phase shifts of the higher-order solitary waves after collision were also found. Numerical solutions were presented for an example where the wave amplitude α was small. This showed evidence of inelastic behavior beyond $O(\alpha)$; an oscillatory wave train of extremely small amplitude was found behind the smaller higher-order solitary wave after the collision. In addition, a good comparison was found between the higher-order phase shifts of the solitary waves and the numerically obtained values.

The aim of this paper is to resolve the differing conclusions of Kraenkel [1] and Marchant and Smyth [5] regarding the nature of the interaction of two higher-order solitary waves. This is done by showing that Krankel's coupling term corresponds to the higher-order phase shift corrections and do not affect the usual scattering properties of the solitons. In Sec. II the asymptotic transformation for the extended KdV equation (1) to the KdV equation is presented. This is similar to the asymptotic transformation used by Kraenkel [1] in that it contains a nonlocal term. A single higher-order solitary wave is derived from the KdV one-soliton solution using the transformation. In Sec. III the KdV two-soliton solution is used to derive the higher-order two-soliton solution for the extended KdV equation (1) with arbitrary coefficients. In addition the higher-order phase shifts are found. In Sec. IV the

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results of Kraenkel [1] are compared with the current theory; it is shown that the coupling term corresponds to the higher-order phase shifts corrections and, hence, both theories are asymptotically equivalent. In Sec. V, it is concluded that the interpretation of the coupling term as a higher-order phase shift is physically appropriate and that the interaction of higher-order solitary waves is asymptotically elastic. This is supported by the numerical evidence of Marchant and Smyth [5].

II. THE ASYMPTOTIC TRANSFORMATION

The asymptotic transformation used in Marchant and Smyth [5] is extended, by including a nonlocal term, to allow the extended KdV equation to be transformed to the KdV equation. This transformation has been recently derived by Fokas and Liu [6] by using the symmetry of the associated higher-order integrable KdV equation. The KdV two-soliton solution is used to directly generate the higher-order two-soliton solution, to $O(\alpha)$, of the extended KdV equation with arbitrary coefficients. If we substitute the transformation

$$\eta = u + \alpha\left(\frac{c_3}{6} - \frac{c_1}{6} + \frac{2c_4}{3}\right)u^2 + \alpha\left(\frac{c_2}{12} - \frac{c_4}{2} - \frac{c_1}{12}\right)u_{xx}, \tag{4}$$

$$\tau = t + \alpha\frac{c_4}{3}x, \quad \xi = x + \alpha c_5 \int_{-\infty}^{x} u(p,t)\,dp, \quad \alpha \ll 1,$$

where $c_5 = (8c_4 - c_3)/3$ and $u(x,t) \to 0$, $x \to \pm\infty$ into the extended KdV equation (1), then $u(\xi,\tau)$ is a solution of the KdV equation

$$u_\tau + 6uu_\xi + u_{\xi\xi\xi} = 0, \tag{5}$$

where terms of $O(\alpha^2)$ are neglected. This transformation is appropriate for solutions that approach zero far upstream and downstream, such as the solitary wave solutions considered here. For other forms of solution, such as periodic solutions, the nonlocal term in the transformation (4) needs to be modified slightly.

There are four perturbative terms at $O(\alpha)$ in Eq. (4), precisely the number of higher-order terms in the extended KdV equation (1). Alternatively, the extended KdV equation (1) can be transformed to a higher-order member of the KdV integrable hierarchy; then only three of the perturbative terms in the transformation (4) are required, as the wave amplitude α can also be rescaled. Marchant and Smyth [5] used the form of Eq. (4) without the nonlocal term while Kraenkel [1] omitted the term perturbing the time t.

The soliton solution of the KdV equation is

$$u = A^* \operatorname{sech}^2\theta \quad \text{where} \quad \theta = k^*(\xi - s - V^*\tau), \tag{6}$$

where A^* is the amplitude, k^* and A^* are related by $2k^{*2} = A^*$, the velocity $V^* = 2A^*$, and s gives the position of the soliton at $t=0$. Using the KdV soliton (6) in the transformation (4) gives

$$\eta = A^* \operatorname{sech}^2\theta + \alpha A^{*2}\left(\frac{c_2}{6} + \frac{c_3}{6} - \frac{c_1}{3} - \frac{c_4}{3}\right)\operatorname{sech}^2\theta + \alpha A^{*2}\left(\frac{3c_4}{2} + \frac{c_1}{4} - \frac{c_2}{4}\right)\operatorname{sech}^4\theta + \cdots, \tag{7}$$

where

$$\theta = k^*\left[\left(1 - \alpha\frac{2c_4}{3}A^*\right)x + \alpha 2k^* c_5(\tanh\theta + 1) - s - V^* t\right].$$

Due to the nonlocal term in the transformation (4), the phase θ is defined implicitly. The phase θ is made explicit by expanding it in a Taylor series. Expanding the phase and scaling the amplitude by using

$$A = A^*[1 - \alpha(4c_4/3)A^*] \tag{8}$$

gives the higher-order solitary wave (7) as

$$\eta = A \operatorname{sech}^2\theta + \alpha A^2 c_6 \operatorname{sech}^2\theta + \alpha A^2 c_7 \operatorname{sech}^4\theta + \cdots,$$

where

$$c_6 = \left(\frac{2c_3}{3} - \frac{c_1}{6} + \frac{c_2}{6} - 5c_4\right), \quad c_7 = \left(\frac{15c_4}{2} - \frac{c_3}{2} + \frac{c_1}{12} - \frac{c_2}{4}\right), \tag{9}$$

$$\theta = k\{x - s[1 + \alpha(2c_4/3)A] + 2\alpha c_5 k - Vt\},$$

$$V = 2A + 4\alpha c_4 A^2 + \cdots,$$

and $2k^2 = A$. Marchant and Smyth [3] derived the higher-order cnoidal wave solution for Eq. (1). Taking the solitary wave limit of this solution [their (2.25)] gives Eq. (9). It should be noted that the transformation has shifted the higher-order solitary wave slightly, from $\xi = s$ at $\tau = 0$ to $x = s(1 + \alpha(2c_4/3)A) + \alpha 2c_5 k$ at $t=0$. Hence, it can be seen that the transformation (4) gives the appropriate expression for a single higher-order solitary wave of the extended KdV equation (1).

III. THE HIGHER-ORDER TWO-SOLITON SOLUTION

The transformation (4) shall be applied to the two-soliton solution of the KdV equation (5) to obtain the corresponding higher-order two-soliton solution, to $O(\alpha)$, for the extended KdV equation (1) with arbitrary coefficients. The two-soliton solution of the KdV equation (5) is

$$\frac{1}{8}u = \frac{k_1^{*2}f_1 + k_2^{*2}f_2 + 2(k_2^* - k_1^*)^2 f_1 f_2 + m(k_2^{*2}f_1^2 f_2 + k_1^{*2}f_1 f_2^2)}{(1 + f_1 + f_2 + m f_1 f_2)^2}, \tag{10}$$

where $f_i = \exp 2k_i^*(V_i^*\tau - \xi + s_i)$, $i=1,2$, $V_i^* = 2A_i^*$, and $m = [(k_2^* - k_1^*)/(k_2^* + k_1^*)]^2$ (see, Hirota in [7]). The velocity of the ith soliton is V_i^* and its position is $s_i + V_i^*\tau$. Well before interaction (as $\tau \to -\infty$,) the two-soliton solution (10) is

$$u = A_1^* \operatorname{sech}^2\theta_1 + A_2^* \operatorname{sech}^2\theta_2, \tag{11}$$

where

$$\theta_i = k_i^*[\xi - s_i - V_i^*\tau], \quad V_i^* = 2A_i^*, \quad i=1,2.$$

Expression (11) is just the sum of two solitons; it satisfies the KdV equation (5) because the solitons are a long distance apart (hence the interaction terms, such as $f_1 f_2$, in Eq. (10) are all negligible). Also we choose $A_1^* > A_2^*$ and $s_1 < s_2$; this means that the larger soliton is behind the smaller one initially. As the first soliton is larger, it travels faster; hence, it will interact with the second soliton as it overtakes it. To see the result of the collision, the solution is considered well after interaction (as $\tau \to \infty$). After interaction, the solution is again Eq. (11), but with the phase shifts

$$+\frac{1}{k_1^*}\ln\left(\frac{k_1^*+k_2^*}{k_1^*-k_2^*}\right), \quad -\frac{1}{k_2^*}\ln\left(\frac{k_1^*+k_2^*}{k_1^*-k_2^*}\right), \tag{12}$$

for the larger and smaller solitons, respectively. Hence, the soliton collision is elastic. The only memory of the collision is a phase shift forwards for the larger soliton and a phase shift backwards for the smaller soliton.

The higher-order two-soliton solution, to $O(\alpha)$, describing the interaction of two higher-order solitary waves governed by the extended KdV equation (1) with arbitrary coefficients, is just the KdV two-soliton solution (10) transformed by using Eq. (4). Due to the complicated form of Eq. (10), the explicit higher-order two-soliton solution will not be calculated; the nature of the collision can be found by considering the solution well before and after interaction. Expression (11) describes the KdV two-soliton solution (10) before and after interaction; substituting Eq. (11) into the transformation (4) gives

$$\begin{aligned}\eta = {} & A_1 \operatorname{sech}^2\theta_1 + A_2 \operatorname{sech}^2\theta_2 + \alpha A_1^2 c_6 \operatorname{sech}^2\theta_1 \\ & + \alpha A_2^2 c_6 \operatorname{sech}^2\theta_2 + \alpha A_1^2 c_7 \operatorname{sech}^4\theta_1 \\ & + \alpha A_2^2 c_7 \operatorname{sech}^4\theta_2 + \cdots,\end{aligned} \tag{13}$$

$$\theta_i = k_i\{x - s_i[1 + \alpha(2c_4/3)A_i] - s_i' - V_i t\},$$

$$V_i = 2A_i + \alpha 4 c_4 A_i^2 + \cdots, \quad i=1,2,$$

where the cross terms (such as $\operatorname{sech}^2\theta_1 \operatorname{sech}^2\theta_2$) are negligible due to the large distance between the solitons. Expression (13) is just two single higher-order solitary waves as required. As for KdV solitons, the higher-order solitary waves are unchanged in shape after the collision. Moreover, no dispersive radiation is generated; hence, the collision is elastic.

The transformation modifies the phase shifts (12), which occur after interaction, of the KdV solitons. Before and after interaction, the phase constants s_i' of each solitary wave are

$$s_1' = \alpha 2 c_5 k_1, \quad s_2' = \alpha 2 c_5 (k_2 + 2k_1), \quad \tau \to -\infty, \tag{14}$$

$$s_1' = \alpha 2 c_5 (k_1 + 2k_2), \quad s_2' = \alpha 2 c_5 k_2, \quad \tau \to \infty.$$

As the nonlocal term in the transformation of the phase θ is an integral from far behind the solitary wave to the current position, an extra term, due to the integration over the trailing solitary wave, can appear in the phase of the leading solitary wave. Hence, it can be seen from Eq. (14) that the extra term appears in the phase of the smaller wave before collision and the larger wave after collision.

There are two contributions to the change in the phase shifts at higher-order, the scaling in amplitude (8) and the extra term that represents the integration over the trailing solitary wave. These phase shifts can be written in the form

$$+\frac{1}{k_1}\ln\left(\frac{k_1+k_2}{k_1-k_2}\right) - 4\alpha\Delta k_2, \quad -\frac{1}{k_2}\ln\left(\frac{k_1+k_2}{k_1-k_2}\right) + 4\alpha\Delta k_1, \tag{15}$$

where

$$\Delta = \tfrac{1}{3}(10c_4 - c_3)$$

for the larger and smaller higher-order solitary waves, respectively. Note that in the special case $c_3 = 10c_4$, the phase shifts are unchanged from the KdV case. These are the same phase shifts as derived by Marchant and Smyth [5] by using a different transformation.

Sachs [8] used a perturbation method based on inverse scattering to examine higher-order solitary wave water interactions. He found that the interactions were elastic to $O(\alpha)$, which is in agreement with the results found above. Zou and Su [9] considered higher-order interactions of solitary waves on shallow water by using a perturbation expansion of the Euler equations. At first-order the solution was assumed to be the KdV two-solition solution. Continuing the perturbation procedure results in partial differential equations describing the solitary wave collision at second- and third-order, which were then solved numerically. At second-order [i.e., at $O(\alpha)$], the solitary wave collision was found to be elastic.

In the special case of surface waves on shallow water, governed by Eq. (1) with coefficients (2), the phase shifts (15) are

$$+\frac{1}{k_1}\ln\left(\frac{k_1+k_2}{k_1-k_2}\right) - 2\alpha k_2, \quad -\frac{1}{k_2}\ln\left(\frac{k_1+k_2}{k_1-k_2}\right) + 2\alpha k_1 \tag{16}$$

for the larger and smaller waves, respectively, which is the same as, after appropriate scalings of space, time, and the amplitude α, those obtained by Zou and Su [9] directly from the Euler water wave equations.

IV. COMPARISON WITH KRAENKEL'S RESULTS

Kraenkel [1] applied the asymptotic transformation to the two-soliton solution of the associated higher-order integrable KdV equation and found higher-order coupling, or interaction, terms which were interpreted as evidence of an inelastic collision. A special case was found where the collisions were elastic, when $c_3=10c_4$. This special case corresponds to when the higher-order corrections in the phase shifts (15) are zero. Here it is shown that the coupling, or interaction, terms of Kraenkel correspond to those higher-order phase shifts.

Firstly, the two-soliton interaction terms of Kraenkel [see Eqs. (14) and (15)] are reproduced. Also Kraenkel's notation is used throughout. The interaction terms are

$$-\frac{\gamma}{2}k_1^4\,\mathrm{sech}^2\left(\frac{\eta_{k_1}}{2}\right)\tanh\left(\frac{\eta_{k_1}}{2}\right)+\frac{\gamma}{2}(k_2^4-2k_1k_2^3)\times\mathrm{sech}^2\left(\frac{\eta_{k_2}+A_{12}}{2}\right)\tanh\left(\frac{\eta_{k_2}+A_{12}}{2}\right),\quad t\to\infty, \tag{17}$$

$$\frac{\gamma}{2}(k_1^4-2k_2k_1^3)\,\mathrm{sech}^2\left(\frac{\eta_{k_1}+A_{12}}{2}\right)\tanh\left(\frac{\eta_{k_1}+A_{12}}{2}\right)-\frac{\gamma}{2}k_2^4\,\mathrm{sech}^2\left(\frac{\eta_{k_2}}{2}\right)\tanh\left(\frac{\eta_{k_2}}{2}\right),\quad t\to-\infty,$$

where the arbitrary phase constant A has been set to zero and k_1 and k_2 identify the two solitary waves and are related to the respective solitary wave amplitudes. Also, the constant A_{12} is related to the phase shifts of the associated higher-order integrable KdV equation. Firstly, if the second solitary wave has zero amplitude ($k_2=0$), then the interaction terms (17) should reduce to the one-soliton perturbative term (see (12) in Kraenkel) for all values of time t. This is not the case in (17) as the remaining perturbative term (with coefficient $\gamma k_1^4/2$) changes sign as time goes from $t\to-\infty$ to $t\to\infty$. Hence, the correct version of Eq. (17) has negative coefficients throughout so that it is consistent with the one-soliton solution.

Consider the KdV soliton

$$w=-\frac{1}{2}k^2\,\mathrm{sech}^2\frac{k}{2}(x+\epsilon s), \tag{18}$$

where s is a higher-order phase shift. As the higher-order phase shift is of $O(\epsilon)$, a Taylor's series expansion gives

$$w=-\frac{1}{2}k^2\,\mathrm{sech}^2\frac{k}{2}(x)+\epsilon s\frac{k^3}{2}\mathrm{sech}^2\frac{k}{2}(x)\tanh\frac{k}{2}(x), \tag{19}$$

where terms of $O(\epsilon^2)$ are ignored. The higher-order term in Eq. (19), which is the Taylor series expansion of the phase shift in Eq. (18), has the same functional form as the interaction terms of Kraenkel. Hence, it seems reasonable to interpret the interaction terms (17) from the two-soliton interaction as phase shifts also. Using this interpretation for the corrected interaction terms (17) gives the phase shifts as

$$s_1=-2\gamma k_2-\gamma k_1,\quad s_2=-\gamma k_2,\quad t\to-\infty, \tag{20}$$

$$s_1=-\gamma k_1,\quad s_2=-2\gamma k_1-\gamma k_2,\quad t\to\infty,$$

well before and after collision. Using Eq. (20) gives the higher-order correction to the phase shifts of the large and small solitary waves as $-2\gamma k_2$ and $2\gamma k_1$, respectively. In terms of the notation used in Sec. III, these corrections are

$$-\tfrac{4}{3}(10c_4-c_3)k_2,\quad \tfrac{4}{3}(10c_4-c_3)k_1, \tag{21}$$

which matches the higher-order phase shifts derived in Eq. (15). Hence, it has been shown that the interaction terms of Kraenkel [1] are asymptotically equivalent to the higher-order corrections to the KdV phase shifts, as found in Sec. III.

V. CONCLUSION

It has been shown that an asymptotic transformation, which includes a nonlocal term, can be used to derive the higher-order two-soliton solution directly from the KdV two-soliton solution. The higher-order solitary waves are elastic to $O(\alpha)$ for all values of the coefficients c_i. Moreover, higher-order corrections to the phase shifts are found. This is in agreement with the theoretical results of other authors, such as Sachs [8], Zou and Su [9], and Marchant and Smyth [5], who also conclude that the interaction is elastic to $O(\alpha)$.

The current theory, and that of Kraenkel [1], have been resolved with Kraenkel's coupling, or interaction, term and the higher-order phase shifts (15) shown to be asymptotically equivalent via a Taylor series expansion. Marchant and Smyth [5] verified the higher-order phase shifts (15) numerically in the special case of surface water waves, which provides additional evidence that the interpretation of the coupling term as a phase shift is physically appropriate.

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